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On Analytic Submanifolds of Different Kahlerian Spaces

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Abstract: The present paper deals with one of two types of submanifolds, namely analytic of certain Kahlerian spaces. Article 1 has been devoted to fundamental results of Kahlerian space whereas in the article 2, we have noted down the results holding good for analytic submanifolds. The articles 3 and 4 deal with totally geodesic analytic submanifolds of symmetric and recurrent Kahlerian spaces respectively and the paper has been concluded by two meaningful remarks.
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1. Preliminaries

Let X_{2n} be a 2n-dimension Kalherian space with F_i^h as structure tensor, g_{ji} as Hermitian metric tensor and ∇ be the operator of covariant differentiation with respect to the christoffel symbols formed with g_{ji}, then

$$(a) \quad F_i^h F_h^j = -\delta_i^j \qquad (b) \quad g_{rs} F_j^r F_i^s = g_{ji} \quad \text{and} \quad (c) \quad \nabla_k F_i^j = 0 \tag{1.1}$$

Here, and in the sequel, the indices i,j,k,... run over the range 1,2,3,...,2n. With the help of Riemannian curvature tensor R_{kji}^h, the covariant curvature tensor $R_{kjih} = g_{jh} R_{kji}^m$, Ricci tensor $R_{ji} = R_{lji}^l = g^{lm} R_{ljim} = g^{lm} R_{jmli}$ and the tensor $S_{ji} = F_j^r R_{ri}, F_{ji} = F_j^h g_{ij}$ etc., we have the expressions

$$P_{kjih} = R_{kjih} + \frac{1}{n+2}[g_{jh}R_{ki} - g_{kh}R_{ji} + F_{ji}S_{kh} - F_{kh}S_{ji} + 2F_{ih}S_{kj}] \tag{1.2}$$

$$\begin{aligned} B_{kjih} = R_{kjih} + \frac{1}{n+4}\{g_{jh}R_{ki} - g_{kh}R_{ji} + g_{ki}R_{jh} - g_{ji}R_{kh} \\ +F_{jh}S_{k1} - F_{kh}S_{ji} + F_{k1}S_{jh} - F_{ji}S_{kh} + 2S_{kj}F_{ih} + 2F_{kj}S_{ih}\} \\ -\frac{R}{(n+2)(n+4)}\{g_{jh}g_{k1} - g_{kh}g_{ji} - F_{jh}F_{ki} - F_{kh}F_{ji} + 2F_{kj}F_{ih}\} \end{aligned} \tag{1.3}$$

and

$$C_{kjih} = R_{kjih} + \frac{1}{n+4}\{g_{jh}R_{ki} - g_{kh}R_{ji} + R_{jh}g_{ki} - R_{kh}g_{ji}$$
$$+F_{ji}S_{ki} - F_{kh}S_{ji} + F_{ki}S_{jh} - F_{ji}S_{kh} + 2S_{kj}F_{ih} + 2F_{kj}S_{ih}\} \quad (1.4)$$

For holomorphically projective curvature tensor (or briefly HP-curvature tensor) ([5] [9]), Bochner curvature tensor [8] and Conharmonic curvature tensor [4]. A tensor field S_{kji} [4] of type (0,3) in Kahlerian space given by

$$S_{kji} = (\nabla_k R_{ji} - \nabla_j R_{ki}) + \frac{1}{n}(g_{ki}\nabla_t R^t_j - g_{ji}\nabla_t R^t_k + F_{ki}\nabla_t S^t_j$$
$$-F_{ji}\nabla_t S^t_k + 2F_{kj}\nabla_t S^t_i) \quad (1.5)$$

will be found useful inward discussion.

2. Analytic Submanifolds:

Let X_{2p} be a 2p-dimensional Riemannian space immersed in X_{2n}, the immersion being given by $x^h = x^h(u^\alpha)$, where $\alpha, \beta, \gamma, ...$ run over the range $1, 2, 3, ..., 2p$. If the transform of tangent space at each point of X_{2n}, by the structure tensor F^h_i of X_{2n} is again tangential to X_{2n}, then the submanifold is called analytic [3] or invariant. For such manifolds, we have

$$g_{ji}B^{ji}_{\alpha\beta} = g_{\alpha\beta} \quad (2.1)$$

$$F^h_i B^i_\alpha = F^\beta_\alpha B^h_\beta \quad (2.2)$$

$$g_{ji}B^j_\alpha C^h_x = 0 \quad (2.3)$$

$$F^h C^i_x = F^y_x C^h_y$$

where $B^i_\alpha = \dfrac{\partial X^i}{\partial U^\alpha}$, $B^{ji}_{\alpha\beta} = B^j_\alpha B^i_\beta$ and C^h_x $(x = 2p+1, ..., 2n)$ are $(2n-2p)$ mutually orthogonal units normals to X_{2p}. The tensor F^α_β induced from F^h_i develops a Kahlerian structure on X_{2p} and satisfies.

$$F^\alpha_\beta F^\beta_\delta = -\delta^\alpha_\delta \quad (2.5)$$

$$g_{\gamma\alpha}F^\gamma_\beta F^\alpha_\delta = g_{\beta\delta}, \quad (2.6)$$

and

$$\nabla_\alpha F^\beta_\gamma = 0 \quad (2.7)$$

where ∇_α denotes the operator of covariant differentiation with respect to the christoffel symbols induced on X_{2p} and are given by

$$\left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} = B_h^\alpha \left(B_\gamma^j B_\beta^i \left\{ \begin{array}{c} h \\ ji \end{array} \right\} + \partial_\gamma B_\beta^h \right) \tag{2.8}$$

where $B_h^\alpha = g^{\alpha\delta} g_{kh} B_\delta^k$ from the equation (2.1)-(2.7), we get,

$$F_{im} B_{\beta\delta}^{im} = F_{\beta\delta}, \quad \text{where} \quad F_{\beta\delta} = F_\beta^\alpha g_{\alpha\delta} \tag{2.9}$$

$$F_{im} B_\alpha^i C_y^m = 0 \tag{2.10}$$

Now, the equations of Gauss [10] and Weingarten [10] are given by

$$\nabla_\alpha B_\beta^h = H_{\alpha\beta x} C_x^h \tag{2.11}$$

and

$$\nabla_\alpha C_x^h = -H_{\alpha x}^\beta B_\beta^h + L_{\alpha xy} C_y^h \tag{2.12}$$

respectively, where $H_{\alpha\beta x}$ are second fundamental tensor of submanifolds with respect to unit normals C_x^h and

$$H_{\gamma x}^\alpha = H g^{\alpha\beta}, L_{\gamma xy} = (\nabla_\gamma C_x^j)(C_\gamma^i g_{ji})$$

For analytic submanifolds, the well known Gauss characteristic equation [3] and the equations of Mainardi-Codazzi [3] are given by

$$R_{kjih} B_{\alpha\beta\gamma\delta}^{kjih} = R_{\alpha\beta\gamma\delta} + H_{\alpha\gamma x} H_{\beta\delta x} - H_{\beta\gamma x} H_{\alpha\delta x} \tag{2.13}$$

and

$$R_{kjih} B_{\alpha\beta\gamma}^{kji} C_x^h = (\nabla_\alpha H_{\beta\gamma x} - \nabla_\beta H_{\alpha\gamma x}) + H_{\beta\gamma z} L_{\alpha zx} - H_{\alpha\gamma z} L_{\beta zx} \tag{2.14}$$

respectively, where $R_{\alpha\beta\gamma\delta}$ are curvature tensor of X_{2p}.

3. Totally geodesic submanifolds of symmetric Kahlerian manifolds

For totally geodesic [10] analytic submanifolds X_{2p}, we have

$$H_{ab}^x = 0 \tag{3.1}$$

and consequently (2.11) and (2.13) reduce into the form

$$\nabla_\alpha B_\beta^h = 0 \tag{3.2}$$

and

$$R_{kjih} B_{\alpha\beta\gamma\delta}^{kjih} = R_{\alpha\beta\gamma\delta} \tag{3.3}$$

respectively. On differentiating (3.3) covariant with respect to (u^α) and using the operator equation $\nabla_\epsilon = B^l_\epsilon \nabla_l$ together with (3.2), we find,

$$\nabla_\epsilon R_{\alpha\beta\gamma\delta} = B^l_\epsilon (\nabla_l R_{kjih}) B^{kjih}_{\alpha\beta\gamma\delta} \tag{3.4}$$

Now, a symmetric Kahlerian space is characterised by [1]

$$\nabla_l R_{kjih} = 0 \tag{3.5}$$

In view of (3.5), (3.4) yields

$$\nabla_\epsilon R_{\alpha\beta\gamma\delta} = 0 \tag{3.6}$$

and so, we have

Theorem (3.1): The totally geodesic analytic submanifolds X_{2p} of a symmetric Kahlerian space X_{2n} is again symmetric.

As an immediate consequence of (3.6), we have

$$\text{(a)} \quad \nabla_\epsilon R_{\alpha\beta} = 0 \qquad \text{(b)} \qquad \nabla_\epsilon R = 0 \tag{3.7}$$

Further, since X_{2p} is also Kahlerian, we have the expression similar to (1.2), (1.3), (1.4) and (1.5) for $P_{\alpha\beta\gamma\delta}, B_{\alpha\beta\gamma\delta}, C_{\alpha\beta\gamma\delta}$ and $S_{\alpha\beta\gamma}$ and consequently, by a straight forward calculation with the help of (2.7), (3.6), (3.7) (a)(b) and $\nabla_\epsilon g_{\alpha\beta} = 0$, we find

$$\nabla_\epsilon P_{\alpha\beta\gamma\delta} = \nabla_\epsilon P_{\alpha\beta\gamma\delta} = \nabla_\epsilon C_{\alpha\beta\gamma\delta} = 0 \tag{3.8}$$

and

$$S_{\alpha\beta\gamma} = 0 \tag{3.9}$$

thus, we have

Theorem (3.2): The totally geodesic analytic submanifolds X_{2p} of a symmetric Kahlerian manifolds X_{2n} is HP symmetric, Bochner symmetric and H-conharmonic symmetric too.

Theorem (3.3): In totally geodesic analytic submanifolds X_{2p} of a symmetric Kahlerian manifolds X_{2n}, the tensor $S_{\alpha\beta\gamma}$ induced from S_{kji} vanishes identically.

4. Totally geodesic submanifolds of recurrent Kahlerian space

Let the Kahlerian space X_{2n} be recurrent [6], then

$$\nabla_l R_{kjih} = k_l R_{kjih} \tag{4.1}$$

where k_l is some non zero vector of recurrence. On substituting from (4.1) into (3.4) and using (3.3), we find

$$\nabla_\varepsilon R_{\alpha\beta\gamma\delta} = (k_l B^l_\varepsilon) R_{\alpha\beta\gamma\delta} \tag{4.2}$$

As k_l are the vectors in X_{2n}, we can write

$$k^h = k^\alpha B^h_\alpha + A_x C^h_x,$$

where k^α is some vector field in X_{2p} and A_x are some e^α functions in the normal space. C multiplying the above equation by $g_{hm}B^m_t$ and using (2.1) and (2.3) we find $k_\varepsilon = k_l B^l_\varepsilon$ at consequently (4.2) yields

$$\nabla_\varepsilon R_{\alpha\beta\gamma\delta} = k_\varepsilon R_{\alpha\beta\gamma\delta}$$

and so, we have

Theorem (4.1): The totally geodesic analytic submanifolds X_{2p} of a recurrent Kahlerian space X_{2n} with k_l as vector of recurrence is again a recurrent space with $k_\varepsilon = k_l B^l_\varepsilon$ vector of recurrence.

As X_{2p} and X_{2n} both are Kahlerian space and it has been proved ([2], [4] and [5]) that Kahlerian recurrent spaces are HP-recurrent, Bochner-recurrent and conharmonic recurrent also with the same vector of recurrence. We conclude the article.

Theorem (4.2): The totally geodesic analytic submanifolds of a recurrent Kahlerian space is,

(i) Kahlerian manifolds with recurrent HP-curvature tensor.

(ii) Kahlerian manifolds with recurrent Bochner curvature tensor.

(iii) Kahlerian manifolds with recurrent H-conharmonic curvature tensor.

Certain Remarks

(a) The theorems proved in articles 3 and 4 can also be proved by writing the Gauss characteristics equations in terms of HP-curvature tensor, Bochner curvature tensor and H-conharmonic curvature tensor.

(b) It has been proved [3] that the variety of analytic subspace of a Kahlerian space is minimal, because

$$g^{\alpha\beta} H_{\alpha\beta\gamma} = 0$$

Holds identically and so the condition $H^x_{\alpha\beta} = g_{\alpha\beta}H^x$ for X_{2p} to be totally umbilical reduces to $H^x_{\alpha\beta} = 0$ which is characterisation of a totally geodesic submanifolds. Thus, "Totally umbilical and Totally geodesic analytic submanifolds of a Kahlerian space are identical".

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